Aspects of Cosmic Inflation in Expanding Bose-Einstein Condensates

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Abstract. Phonons in expanding Bose-Einstein condensates with wavelengths much larger than the healing length behave in the same way as quantum fields within a universe undergoing an accelerated expansion. This analogy facilitates the application of many tools and concepts known from general relativity (such as horizons) and the prediction of the corresponding effects such as the freezing of modes after horizon crossing and the associated amplification of quantum fluctuations. Basically the same amplification mechanism is (according to our standard model of cosmology) supposed to be responsible for the generation of the initial inhomogeneities – and hence the seeds for the formation of structures such as our galaxy – during cosmic inflation (i.e., a very early epoch in the evolution of our universe). After a general discussion of the analogy (*analogue cosmology*), we calculate the frozen and amplified density-density fluctuations for quasi-two dimensional (Q2D) and three dimensional (3D) condensates which undergo a free expansion after switching off the (longitudinal) trap.

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1. Introduction

A quarter of a century ago, W. G. Unruh [1] noticed an intriguing analogy between (at a first glance) very different systems: Phonons in irrotational fluids behave in the same way as (quantum) fields in curved space-times whose geometry is determined by the effective metric and depends on the velocity of the fluid flow and the speed of sound etc. This analogy facilitates a know-how transfer in both directions: On the one hand, it allows us to apply all the concepts, tools, and effects known from general relativity – such as horizons – to the propagation of phonons (or other quasi-particles) in fluids and hence leads to a better understanding of condensed-matter phenomena in terms of universal geometrical concepts. On the other hand, this analogy opens up the opportunity of a theoretical and possibly experimental investigation of exotic quantum effects known from cosmology such as Hawking radiation [2].

Unfortunately, it turns out that an experimental verification of the Hawking effect by means of such an analogue system (i.e., a black hole analogue) is very difficult [3]. In this article, we shall focus on another exotic quantum phenomenon known from cosmology which also involves the amplification of the initial quantum vacuum fluctuations due to the presence of a horizon: Within our standard model of cosmology, the early universe was nearly homogeneous and basically all inhomogeneities – including the seeds for the formation of large-scales structures such as our galaxy – originate from the quantum vacuum fluctuations of a scalar field, which is called the inflaton. During inflation ([4]), which is a (conjectured) period of accelerated expansion of the very early universe, these quantum fluctuations were amplified due to the presence of a cosmic horizon (cf. Section 5). These amplified fluctuations left an imprint in the cosmic microwave background radiation – small perturbations upon a homogeneous background as measured by the WMAP satellite [5].

Owing to the aforementioned analogy, the very same amplification mechanism should also occur in appropriate fluids. As an example, we shall consider the in some sense best understood superfluids – dilute atomic/molecular Bose-Einstein condensates (BECs), see, e.g., [6] for review. Apart from the good theoretical understanding, further advantages of BECs are the various possibilities to manipulate (e.g., changing the speed of sound) and control them experimentally. As it will become evident later, the free expansion of a BEC after switching off the trap – which is a standard procedure in time-of-flight experiments/measurements, for example – leads to the formation of an effective cosmic horizon and, consequently, to the amplification of the initial quantum fluctuations of the phonon modes. These amplified fluctuations manifest themselves in potentially measurable small-scale density (and phase) variations.

In order to study this effect for rather general conditions, we shall consider BECs with non-specified power-law self-interactions in an arbitrary number of (spatial) dimensions. After a discussion of the effectively lower-dimensional behaviour of strongly constrained BECs in Section 2, the effective metric governing the propagation of phonons is derived in Section 3 for arbitrary power-law self-interactions in any dimension.

Section 4 is devoted to the introduction of co-moving coordinates (in complete analogy to cosmology) and to the scaling behaviour of the expanding condensate. The analogy to cosmology is further elaborated in Section 5 by applying the concept of an effective horizon, whose existence generates the aforementioned amplification mechanism. The spectrum and magnitude of the resulting density fluctuations are calculated in Sections 6 and 7 for the quasi-two and the three-dimensional case, respectively, assuming the usual quartic coupling.

2. Dimensional Reduction

Since we shall consider an arbitrary number of spatial dimensions later on, let us first discuss the behaviour of strongly confined and hence effectively lower-dimensional Bose-Einstein condensates. In three spatial dimensions, Bose-Einstein condensates are described by the Lagrange density ($\hbar = 1$ throughout) [7]

$$\mathcal{L} = \frac{i}{2}(\Psi^*\dot{\Psi} - \dot{\Psi}^*\Psi) - \frac{|\boldsymbol{\nabla}\Psi|^2}{2m} - V_{\rm ext}(\boldsymbol{r},t)|\Psi|^2 - \frac{g}{2}\,|\Psi|^4\,, \tag{1}$$

with m being the mass of the bosons, $V_{\rm ext}$ the external and generally space-time dependent one-particle trapping potential, and the two-particle coupling g in s-wave approximation, which is related to the s-wave scattering length a_s via $g = 4\pi a_s/m$. If the external one-particle trapping potential can be split up into a parallel and a static transversal part via

$$V_{\rm ext}(\boldsymbol{r},t) = V_{\rm ext}^{\parallel}(\boldsymbol{r}_\parallel,t) + V_{\rm ext}^{\perp}(\boldsymbol{r}_\perp)\,, \tag{2}$$

it is useful to decompose the order parameter Ψ (and its quantum fluctuations $\delta\hat{\Psi}$) into a complete set of real and time-independent functions $\phi_\alpha(\boldsymbol{r}_\perp)$ governing the transversal dependence (cf. [8] where the density of an elongated BEC is decomposed in a similar manner)

$$\Psi(\boldsymbol{r},t) = \sum_\alpha \phi_\alpha(\boldsymbol{r}_\perp)\psi_\alpha(\boldsymbol{r}_\parallel,t)\,. \tag{3}$$

The time evolution of the coefficient functions $\psi_\alpha(\boldsymbol{r}_\parallel,t)$ is determined by the reduced Lagrangian density

$$\begin{aligned}
\mathcal{L}_\parallel &= \int dV_\perp^{3-D}\,\mathcal{L} \\
&= \sum_{\alpha\beta}\left[\frac{i}{2}(\psi_\alpha^*\dot{\psi}_\beta - \dot{\psi}_\alpha^*\psi_\beta) - \frac{1}{2m}(\boldsymbol{\nabla}\psi_\alpha^*)\cdot(\boldsymbol{\nabla}\psi_\beta) - V_{\rm ext}^{\parallel}\psi_\alpha^*\psi_\beta\right]\int dV_\perp^{3-D}\,\phi_\alpha\phi_\beta \\
&\quad - \sum_{\alpha\beta}\psi_\alpha^*\psi_\beta\int dV_\perp^{3-D}\,\phi_\alpha\left(-\frac{\boldsymbol{\nabla}_\perp^2}{2m} + V_{\rm ext}^{\perp}\right)\phi_\beta \\
&\quad - \frac{g}{2}\sum_{\alpha\beta\gamma\delta}\psi_\alpha^*\psi_\beta\psi_\gamma^*\psi_\delta\int dV_\perp^{3-D}\,\phi_\alpha^*\phi_\beta\phi_\gamma^*\phi_\delta\,.
\end{aligned} \tag{4}$$

If we now choose the functions $\phi_\alpha(\boldsymbol{r}_\perp)$ to be the orthonormal eigenfunctions of the self-adjoint operator $\mathcal{K}$ with eigenvalues Ω_α

$$\mathcal{K}\phi_\alpha = \left(-\frac{\boldsymbol{\nabla}_\perp^2}{2m} + V_{\rm ext}^{\perp}\right)\phi_\alpha = \Omega_\alpha\phi_\alpha\,, \tag{5}$$

the reduced Lagrangian density simplifies to

$$\mathcal{L}_\parallel = \sum_\alpha \left[\frac{i}{2}(\psi_\alpha^*\dot{\psi}_\alpha - \dot{\psi}_\alpha^*\psi_\alpha) - \frac{1}{2m}|\boldsymbol{\nabla}\psi_\alpha|^2 - (V_{\rm ext}^\parallel + \Omega_\alpha)|\psi_\alpha|^2\right]$$
$$- \frac{g}{2}\sum_{\alpha\beta\gamma\delta}\psi_\alpha^*\psi_\beta\psi_\gamma^*\psi_\delta \int dV_\perp^{3-D}\,\phi_\alpha^*\phi_\beta\phi_\gamma^*\phi_\delta\,. \tag{6}$$

The last term induces a coupling of different modes, which complicates the solution. However, if we assume that the coupling term is sufficiently small and the lowest mode $\alpha = 0$ dominates $|\psi_0| \gg |\psi_{\alpha>0}|$, we may estimate the population of the higher modes in analogy to stationary perturbation theory: The mixing between the lowest mode $\alpha = 0$ and all higher modes $\alpha > 0$ is small if the energy differences $\Delta\Omega_\alpha = \Omega_\alpha - \Omega_0$ are large compared to the transition matrix elements of the interaction Hamiltonian, i.e.,

$$\Delta\Omega_\alpha \gg g|\psi_0|^2 \int dV_\perp^{3-D}\,\phi_\alpha^*\phi_\beta\phi_\gamma^*\phi_\delta\,. \tag{7}$$

Since the modes ϕ_α are normalised $\phi_\alpha \sim 1/\sqrt{V_\perp^{3-D}}$, this condition can be re-expressed in terms of the size of the transversal dimension $a_\perp$ and the healing length $\xi = 1/\sqrt{g|\psi_0|^2 m}$. Therefore, imposing the conditions

$$\xi \gg a_\perp \gg a_s\,, \tag{8}$$

where the latter requirement $a_\perp \gg a_s$ is necessary for the Gross-Pitaevskiĭ Lagrangian density in Eq. (1) to be valid, we arrive at the reduced Lagrangian density for the lowest mode $\psi = \psi_0$

$$\mathcal{L}_\parallel = \frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{1}{2m}|\boldsymbol{\nabla}\psi|^2 - V_{\rm ext}^\parallel|\psi|^2 - \frac{g_\parallel}{2}\,|\psi|^4\,, \tag{9}$$

where Ω_0 has been absorbed into $V_{\rm ext}^\parallel$ and

$$g_\parallel = g\int dV_\perp^{3-D}\,|\phi_0|^4 = \frac{4\pi a_s}{m}\int dV_\perp^{3-D}\,|\phi_0|^4 \propto \frac{a_s}{a_\perp^{3-D}}\,, \tag{10}$$

denotes the reduced (lower-dimensional) coupling constant. For example, the eigenfunctions $\phi_\alpha(\boldsymbol{r}_\perp) = \phi_\alpha(z)$ of a harmonic potential $V_\perp(z) = m\omega_z^2 z^2/2$ with the transversal length scale $a_\perp = 1/\sqrt{m\omega_z}$ are just the Legendre polynomials with an equidistant energy spectrum. In this case, the reduced coupling is given by $g_\parallel = g\sqrt{m\omega_z/2\pi}$. However, it should be emphasised that the method presented above is applicable to more general potentials satisfying the aforementioned conditions as well.

Note that if we relaxed condition (8) and the transverse size of the condensate, a_z, would be of comparable to the scattering length, $a_s/a_z = \mathcal{O}(1)$, we would obtain a lower-dimensional condensate with different interactions due to density-dependent corrections to the coupling coefficient [9, 10, 11, 12].

3. Effective Geometry

After having discussed the dimensional reduction, let us start with the Lagrange density in an arbitrary number of spatial dimensions D (where we omit the superscript $\parallel$ for

the sake of conciseness). In addition, we shall assume a more general self-coupling term $|\psi|^{2N}$ which will be motivated later

$$\mathcal{L} = \frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \frac{1}{2m}|\boldsymbol{\nabla}\psi|^2 - V_{\rm ext}|\psi|^2 - \frac{g}{2}\,|\psi|^{2N}\,. \tag{11}$$

Inserting the Madelung representation

$$\psi = \sqrt{\varrho}\,e^{iS}\,, \tag{12}$$

the Lagrange density reads

$$\mathcal{L} = -\varrho\,\partial_t S - \frac{\varrho}{2m}(\boldsymbol{\nabla}S)^2 - \frac{(\boldsymbol{\nabla}\sqrt{\varrho})^2}{2m} - V_{\rm ext}\,\varrho - \frac{g}{2}\,\varrho^N\,. \tag{13}$$

As usual, variation w.r.t. S yields the equation of continuity and w.r.t. ϱ the Bernoulli equation with the mean-field velocity $\boldsymbol{v} = \boldsymbol{\nabla}S/m$ and the specific pressure $p(\varrho, \boldsymbol{\nabla}^2\sqrt{\varrho})$. Assuming that the density profile is sufficiently smooth, i.e., that the relevant length scales are much larger than the healing length (Thomas-Fermi approximation), we neglect the quantum pressure term $(\boldsymbol{\nabla}\sqrt{\varrho})^2$. Linearisation $S = S_0 + \delta S$ and $\varrho = \varrho_0 + \delta\varrho$ around a background solution S_0 and ϱ_0 yields the second-order Lagrange density (cf. [13, 14, 15, 16])

$$\mathcal{L}^{(2)} = -\delta\varrho\,\partial_t\delta S - \frac{\varrho_0}{2m}(\boldsymbol{\nabla}\delta S)^2 - \delta\varrho\,\boldsymbol{v}_0\cdot\boldsymbol{\nabla}\delta S - \frac{g_N}{2}\,\delta\varrho^2\,, \tag{14}$$

where we have introduced the effective coupling

$$g_N = g\,\frac{N(N-1)}{2}\,\varrho_0^{N-2}\,. \tag{15}$$

For the usual case $N = 2$, we have $g_N = g$, but one should still bear in mind that we are considering an arbitrary number of spatial dimensions D. Note that there are no sound waves at all for $N = 1$, since the theory is non-interacting in that case.

In the Thomas-Fermi approximation, the linearised Bernoulli equation can be solved for the density fluctuations

$$\delta\varrho = -\frac{\partial_t + \boldsymbol{v}_0\cdot\boldsymbol{\nabla}}{g_N}\,\delta S\,, \tag{16}$$

and inserting this result back into Eq. (14), we obtain the effective Lagrangian for the phase fluctuations $\phi = \delta S$ only

$$\mathcal{L}^{(2)}_{\rm eff} = \frac{1}{2g_N}\left(\dot{\phi} + \boldsymbol{v}_0\cdot\boldsymbol{\nabla}\phi\right)^2 - \frac{\varrho_0}{2m}(\boldsymbol{\nabla}\phi)^2\,. \tag{17}$$

Even in an arbitrary number of spatial dimensions D (the difficulties for $D = 1$ will be discussed below) and for general self-coupling $N > 1$, this Lagrangian is completely equivalent to that of a free (minimally coupled) scalar field in a curved space-time (e.g., [17])

$$\mathcal{L}^{(2)}_{\rm eff} = \frac{1}{2}\sqrt{|\mathrm{g}_{\rm eff}|}\,(\partial_\mu\phi)\mathrm{g}^{\mu\nu}_{\rm eff}(\partial_\nu\phi)\,, \tag{18}$$

provided that we insert the Painlevé-Gullstrand-Lemaître (PGL) metric [18]

$$\mathrm{g}^{\rm eff}_{\mu\nu} = A_D^{(N)}\begin{pmatrix} c_N^2 - \boldsymbol{v}_0^2 & \boldsymbol{v}_0 \\ \boldsymbol{v}_0 & -\mathbf{1}\end{pmatrix}\,,$$

$$\mathrm{g}^{\mu\nu}_{\rm eff} = \frac{1}{A_D^{(N)}c_N^2}\begin{pmatrix} 1 & \boldsymbol{v}_0 \\ \boldsymbol{v}_0 & \boldsymbol{v}_0\otimes\boldsymbol{v}_0 - c_N^2\mathbf{1}\end{pmatrix}\,, \tag{19}$$

with the speed of sound $c_N^2 = g_N \varrho_0/m$ and the conformal factor

$$A_D^{(N)} = \left(\frac{c_N}{g_N}\right)^{2/(D-1)} . \tag{20}$$

This expression already indicates problems in one spatial dimension $D = 1$ due to conformal invariance of the scalar field action (18) in 1+1 dimensions. Without the introduction of an additional dilaton field, the identification of an effective metric is only possible if $c_N/g_N = \text{const}$, for example if $g = \text{const}$ and $N = 3$, see also the next Section.

4. Co-Moving Coordinates and Scaling

Since phase fluctuations in Bose-Einstein condensates behave (in the Thomas-Fermi approximation) exactly as a scalar field in a specific curved space-time described by the effective metric in Eq. (19), it will be useful to investigate this metric with the tool known from general relativity. The effective line element reads

$$ds^2_{\text{eff}} = A_D^{(N)} \left([c_N^2 - \boldsymbol{v}_0^2]dt^2 + 2\boldsymbol{v}_0 \cdot d\boldsymbol{r}dt - d\boldsymbol{r}^2\right) . \tag{21}$$

In analogy to cosmology, we shall assume local isotropy and homogeneity – which will be a good approximation in the centre of the BEC cloud. Since the coupling g is supposed to be constant for simplicity, this assumption implies a spatially homogeneous but possibly time-dependent density $\varrho_0 = \varrho_0(t)$ and effective coupling $g_N = g_N(t)$. Furthermore, after a suitable re-definition of the origin of our coordinate system, we may set $\boldsymbol{v}_0 \propto \boldsymbol{r}$ due to the presumed local isotropy and homogeneity. Insertion of this *ansatz* into the equation of continuity yields ([14, 15])

$$\varrho_0(t) = \frac{\varrho_0(t=0)}{b^D(t)} \leftrightarrow \boldsymbol{v}_0(t,\boldsymbol{r}) = \frac{\dot{b}}{b}\boldsymbol{r} , \tag{22}$$

which allows us to describe $\varrho_0(t)$ and $\boldsymbol{v}_0(t,\boldsymbol{r})$ as well as $g_N(t)$ by means of a single time-dependent scaling parameter $b(t)$ satisfying the initial condition $b(t=0) = 1$ and $\dot{b}(t=0) = 0$. As we know from general relativity, an off-diagonal metric such as in Eq. (21) can be diagonalised by introducing co-moving spatial coordinates via (cf. the scaling transformation for BECs in [19, 20])

$$\boldsymbol{\rho} = \frac{\boldsymbol{r}}{b(t)} \to d\boldsymbol{\rho} = \frac{d\boldsymbol{r} - \boldsymbol{v}_0 dt}{b(t)} \to ds^2_{\text{eff}} = A_D^{(N)}\left(c_N^2 dt^2 - b^2 d\boldsymbol{\rho}^2\right) . \tag{23}$$

In addition, we may transform from the laboratory time t to the effective proper (co-moving wrist-watch) time τ in order to eliminate the factor in front of $d\tau^2$

$$\tau = \int dt \sqrt{A_D^{(N)}}\, c_N \to ds^2_{\text{eff}} = d\tau^2 - A_D^{(N)} b^2 d\boldsymbol{\rho}^2 , \tag{24}$$

arriving at the standard Friedmann-Robertson-Walker representation, see, e.g., [17]. Let us investigate the remaining factor $A_D^{(N)}b^2$ a bit further: Since $\varrho_0(t) \propto b^{-D}(t)$ according to Eq. (22), we obtain $g_N(t) \propto b^{-D(N-2)}(t)$ from Eq. (15) as well as $c_N(t) \propto b^{-D(N-1)/2}(t)$, i.e., the factor in front of $d\boldsymbol{\rho}^2$ scales as

$$A_D^{(N)} b^2 \propto b^{2+D(N-3)/(D-1)} , \tag{25}$$

according to Eq. (20). Interestingly, the exponent vanishes for

$$N = \frac{2}{D} + 1\,, \tag{26}$$

e.g., for $D = 2$ and $N = 2$, or $D = 3$ and $N = 5/3$, or $D = 1$ and $N = 3$. As we have observed above, special care is required for the latter case $D = 1$, but for $N = 3$ we can indeed introduce an effective metric and the conformal factor can be chosen at will since it does not enter the calculation.

If the condition in Eq. (26) is satisfied and thus the exponent in Eq. (25) vanishes, the transformed metric in Eq. (24) is flat and hence the wave equation for the phonon modes becomes trivial in terms of the coordinates τ and $\boldsymbol{\rho}$

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{1}{A_D^{(N)}(t=0)}\frac{\partial^2}{\partial\boldsymbol{\rho}^2}\right)\phi = 0\,, \tag{27}$$

i.e., a solution $\phi(\tau, \boldsymbol{\rho})$ is independent of the external time dependence mediated via the scaling parameter $b(t)$. It turns out that this perfect scaling is not only valid for the phonon modes, but can be extended to the full field operator: If we start from the equation of motion in the Heisenberg picture

$$i\frac{\partial}{\partial t}\hat{\Psi} = \left(-\frac{1}{2m}\frac{\partial^2}{\partial\boldsymbol{r}^2} + V_{\rm ext}(t, \boldsymbol{r}) + g\,(\hat{\Psi}^\dagger)^{N-1}\hat{\Psi}^{N-1}\right)\hat{\Psi}\,, \tag{28}$$

again with general s-wave coupling (N) and in D spatial dimensions, we may account for an arbitrarily time-dependent external one-particle trapping potential $V_{\rm ext}(t, \boldsymbol{r})$ as long as it is purely harmonic at all times

$$V_{\rm ext}(t, \boldsymbol{r}) = \frac{m}{2}\,\omega^2_{\rm ext}(t)\boldsymbol{r}^2\,, \tag{29}$$

by inserting the scaling *ansatz* derived above (cf., [20])

$$\hat{\Psi}(t, \boldsymbol{r}) = \frac{\exp\{im\boldsymbol{v}_0^2(t, \boldsymbol{r})/2\}}{\sqrt{b^D(t)}}\,\hat{\psi}(\tau, \boldsymbol{\rho})\,. \tag{30}$$

The numerator $\exp\{im\boldsymbol{v}_0^2(t, \boldsymbol{r})/2\}$ of the pre-factor reproduces $\boldsymbol{v}_0$ and the denominator $\sqrt{b^D(t)}$ accounts for $\varrho_0(t) = \varrho_0(t=0)/b^D(t)$ and ensures the correct commutation relations for $\hat{\psi}(\tau, \boldsymbol{\rho})$, i.e.,

$$[\hat{\psi}(\tau, \boldsymbol{\rho}), \hat{\psi}^\dagger(\tau, \boldsymbol{\rho}')] = \delta^D(\boldsymbol{\rho} - \boldsymbol{\rho}')\,. \tag{31}$$

For a flat metric $N = 1 + 2/D$, the proper time is (independently of D) determined as

$$\tau = \int \frac{dt}{b^2(t)}\,, \tag{32}$$

and $\boldsymbol{\rho}$ is of course still given by $\boldsymbol{\rho} = \boldsymbol{r}/b(t)$. Finally, if we arrange the scale parameter $b(t)$ according to

$$\ddot{b}(t) + \omega^2_{\rm ext}(t)b(t) = \frac{\omega^2_{\rm ext}(t=0)}{b^3(t)} = \frac{\omega_0^2}{b^3(t)}\,, \tag{33}$$

the equation of motion for $\hat{\psi}(\tau, \boldsymbol{\rho})$ becomes independent of $b(t)$

$$i\frac{\partial}{\partial\tau}\hat{\psi}(\tau, \boldsymbol{\rho}) = \left(-\frac{1}{2m}\frac{\partial^2}{\partial\boldsymbol{\rho}^2} + \frac{m}{2}\,\omega_0^2\boldsymbol{\rho}^2 + g\,(\hat{\psi}^\dagger)^{N-1}\hat{\psi}^{N-1}\right)\hat{\psi}(\tau, \boldsymbol{\rho})\,, \tag{34}$$

i.e., we obtain a perfect scaling solution of the full field operator exactly in those cases where the effective metric is flat. Note that the implications of this property of the full field operator go far beyond the scaling of the hydrodynamic solution – we also obtain a perfect scaling of the quantum fluctuations to arbitrary order and for large wavenumbers (provided that the s-wave approximation is still valid, of course), where hydrodynamic (Thomas-Fermi) solution breaks down. It is not even necessary to assume the mean-field expansion, i.e., the scaling also applies to non-condensed bosons.

Interestingly, all three examples (from $D = 1$ to $D = 3$) are potentially relevant for real physical systems:

- $D = 1$ and $N = 3$: In quasi-1D Bose-Einstein condensates, one may obtain an effective $|\psi|^6$ coupling if the perpendicular size a_z is comparable to the s-wave scattering length a_s, see, e.g., [10, 11, 12].
- $D = 2$ and $N = 2$: This is the usual quasi-2D Bose-Einstein condensate with quartic coupling, e.g., [6].
- $D = 3$ and $N = 5/3$: Even though the exponent $N = 5/3$ may seem somewhat unnatural, it appears in an effective description of a weakly interacting two-component Fermi gas in the BCS state, where the equation of state is determined by the Fermi energy $E_{\mathrm{F}} \propto \varrho^{5/3}$, and which therefore also shows scaling behaviour, see, e.g., [14].

5. Horizon Analogues

Apart from the question of whether it can be cast into a flat space-time form by means of an appropriate coordinate transformation or not, the emergence of a non-trivial effective metric in Eq. (19) suggests the application of concepts known from general relativity, such as horizons [4, 21, 22]. Generally speaking, horizons correspond to a loss of causal connectivity, i.e., events beyond a horizon have no influence or cannot be influenced. The most prominent example is the event horizon of a (classical) black hole, beyond which everything is trapped forever (i.e., nothing can ever escape to infinity).

On the other hand, in cosmology, two slightly different horizon concepts play a more important role due to the large-scale homogeneity and isotropy – the particle horizon and the apparent horizon. The particle horizon always refers to a chosen trajectory and indicates the border to the space-time region which cannot be reached by any signal starting from this trajectory or from where no signal can reach this trajectory. The apparent horizon depends on the chosen coordinates and is (roughly speaking) defined as the border beyond which all closed two-surfaces can either only expand or only contract w.r.t. the chosen coordinates.

As it turns out, the concepts of the particle and the apparent horizon can be applied to expanding Bose-Einstein condensates. Let us start with the particle horizon and choose the trajectory $\boldsymbol{r} = 0 \to \boldsymbol{\rho} = 0$ for which the particle horizon can conveniently be determined using the metric in Eq. (23). The question is: which values of the co-moving coordinate $\boldsymbol{\rho}$ can a sound wave reach by starting at the origin $\boldsymbol{\rho} = 0$ at time

t and being described by a null line $ds^2_{\rm eff} = 0$, i.e., how far can it travel? According to Eq. (23), this length can be calculated via the following integral (cf. also [4])

$$\Delta\rho_{\rm horizon}(t) = \int\limits_t^\infty dt' \, \frac{c_N(t')}{b(t')} = c_N(t=0) \int\limits_t^\infty \frac{dt'}{b^{1+D(N-1)/2}(t')} \,. \tag{35}$$

If the above integral converges to a finite result $\Delta\rho_{\rm horizon}(t)$, this corresponds to a particle horizon since no point beyond this co-moving coordinate can be reached by sound waves anymore.

For a free expansion of the condensate $V_{\rm ext}(t \uparrow \infty) = 0$, we obtain $b(t \uparrow \infty) \propto t$ at late times according to Eq. (33) and thus a horizon exists for all $N > 1$. The occurrence of a horizon might be puzzling if the metric in terms of the proper time τ in Eq. (24) is flat – but this puzzle can be resolved by the observation that the proper time τ reaches a finite value $\tau(t \uparrow \infty) < \infty$ in the limit of arbitrarily late laboratory times in that case, cf. Eq. (32).

In contrast to the particle horizon, which necessitates the knowledge of the full future (or past), the apparent horizon can be identified by means of the configuration at a certain instant of time only. Instead of a trajectory as for the particle horizon, we have to choose a specific coordinate system (time slices) in order to define the apparent horizon. The coordinates $(\tau, \boldsymbol{\rho})$ leading to a flat metric, for example, do of course not allow the introduction of an apparent horizon. On the other hand, since we observe the expanding condensate using the laboratory coordinate system $(t, \boldsymbol{r})$, we shall choose these coordinates instead. For a spherically symmetric metric as in Eq. (19), the apparent horizon is determined by $g^{\rm eff}_{00} = 0$, i.e., where the velocity of the condensate $\boldsymbol{v}_0$ exceeds the speed of sound c_N

$$\boldsymbol{v}_0^2(t, r_{\rm horizon}) = c_N^2(t) \;\to\; r_{\rm horizon}(t) \propto \frac{b^{1-D(N-1)/2}}{\dot b} \,. \tag{36}$$

The very intuitive interpretation is that no sound wave can enter the region $r < r_{\rm horizon}$ from the outside.

For perfect scaling $N = 1 + 2/D$ and a freely expanding condensate, the apparent horizon settles down at late times (when $\dot b$ becomes constant) to a finite value $r_{\rm horizon}(t \uparrow \infty) = r^\infty_{\rm horizon} > 0$, otherwise it may increase or decrease forever (depending on D and N). Note that both, the apparent and the particle horizon are never at rest w.r.t. the co-moving coordinate $\boldsymbol{\rho}$ but always decreasing. As a result, the wavelength of all sound modes $\exp\{i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}\}$ exceeds the horizon size at some point of time (horizon crossing) in view of the permanent stretching of modes due to the expansion of the condensate. After the wavelength exceeds the horizon size, the regions of higher and lower pressure cannot interact anymore and hence the modes stop oscillating (freezing of modes). As a result, comparing each mode with a harmonic oscillator, the momentum and its variance decrease drastically – and according to the Heisenberg uncertainty relation, the complementary variance must increase in order to compensate this. Ergo, the initial ground/vacuum state gets squeezed, which amplifies the quantum fluctuation in a certain direction. This admittedly rather intuitive picture (horizon crossing,

freezing, and squeezing) applies to both, the expanding universe (provided a horizon exists, such as during cosmic inflation) as well as expansing Bose-Einstein condensates in a very similar way and must of course be further supported by explicit calculations, see the next Section.

Note that analogue horizons (and perhaps horizons in real gravity as well) are only low-energy effective concepts since, for very small wavelengths (below the healing length), the quantum pressure term becomes important leading to group and phase velocities exceeding the usual speed of sound. Furthermore, as the healing length increases during the expansion, the late-time limit in the derivation of the particle horizon is not strictly valid. Nevertheless, the main effects such as the amplification of quantum fluctuations are not modified drastically, see the next Section.

6. Density-Density Correlations in quasi-2D

According to the results of Section 4, the time-evolution of a quasi-two-dimensional condensate with the usual quartic coupling (i.e., $N = D = 2$) is particularly simple to solve due to its perfect scaling behaviour – which enables us to derive its dynamics exactly (e.g., including the quantum-pressure corrections). If we assume that the initial trapping potential is harmonic, the free expansion after switching off the longitudinal trap $V_{\|}^{\text{ext}}(t > 0) = 0$ is fully described by the time-evolution of the scaling parameter according to Eq. (33)

$$b(t) = \sqrt{1 + \omega_0^2 t^2}\,. \tag{37}$$

The associated proper time τ can be calculated from Eq. (32)

$$\tau(t > 0) = \frac{\arctan \omega_0 t}{\omega_0}\,. \tag{38}$$

Since τ quickly approaches a finite value at late times $t \uparrow \infty$, the fluctuations freeze in and assume the value at proper time $\tau = \pi/(2\omega_0)$. In view of the perfect scaling discussed in Sec. 4, both the background and the phonon modes – expressed in terms of $\boldsymbol{\rho}$ and τ – behave in the same way before and during the expansion, leaving the relative correlation function

$$C(\boldsymbol{\rho}, \boldsymbol{\rho}') = \frac{\langle \delta\hat{\varrho}(t, \boldsymbol{\rho})\delta\hat{\varrho}(t, \boldsymbol{\rho}')\rangle}{\varrho_0(t, \boldsymbol{\rho})\varrho_0(t, \boldsymbol{\rho}')} = \frac{\langle \delta\hat{\varrho}(\boldsymbol{\rho})\delta\hat{\varrho}(\boldsymbol{\rho}')\rangle}{\varrho_0(\boldsymbol{\rho})\varrho_0(\boldsymbol{\rho}')} \tag{39}$$

unchanged. Consequently, the spectrum of the relative density contrast in co-moving coordinates after horizon crossing is given by the initial correlation spectrum. Besides the quasi-2D condensate with usual quartic coupling ($N = D = 2$), this also holds for all other cases where perfect scaling occurs.

As discussed in Section 5, the apparent horizon depends on the chosen time-slicing, i.e., the set of coordinates. (The particle horizon is independent of the set of coordinates, but requires the knowledge of the whole future evolution.) Co-moving coordinates, albeit advantageous to describe the evolution of the modes, are not suitable for describing the appearance of an apparent horizon. On the other hand, all observations are performed

in laboratory time t and coordinates $\boldsymbol{r}$ with the metric (19) showing the formation of an apparent horizon according to Eq. (36). For $N = 1 + 2/D$, it follows

$$r_{\text{horizon}}(t) = \frac{c_N(0)}{\dot{b}(t)} = c_N(0)\frac{\sqrt{1+\omega_0^2 t^2}}{\omega_0^2 t}\,, \tag{40}$$

in the homogeneous case. While the trapping potential is still turned on ($t < 0$), the (apparent) horizon size r_{horizon} is infinite. Clearly, without any expansion, all points are causally connected via sound waves. After releasing the condensate, however, the horizon settles down very quickly ($\omega_0 t \gg 1$) at a finite position (in laboratory coordinates) $c_N(0)/\omega_0$, given by the initial speed of sound $c_N(0)$ and trapping frequency ω_0.

Since the (apparent) horizon approaches a finte position in laboratory coordinates $(t, \boldsymbol{r})$ and every phonon mode with a given wavelength λ expands with the condensate cloud, each phonon mode will cross the horizon eventually. Equivalently, in terms of the co-moving coordinates, the horizon size decreases for all times (as long as the condensate is expanding) and the (co-moving) wavenumber $\boldsymbol{\kappa}$ remains constant. As mentioned before, the initial correlation function of a quasi-2D condensate translates directly into the frozen density contrast after horizon crossing.

In order to derive the density-density correlation function quantitatively, we quantise the phonon modes within the initial condensate $\boldsymbol{v}_0 = 0$ using the approximation (centre of the BEC cloud) of a constant background density $\varrho_0 = \text{const}$. After a normal mode expansion, the Lagrange function (13) reads

$$L = \int d^2 r\,\mathcal{L} = \frac{V_Q}{4}\sum_{\boldsymbol{k}}\left[\left(g_{2\text{D}} + \frac{\boldsymbol{k}^2}{4m\varrho_0}\right)^{-1}(\delta\dot{S}_{\boldsymbol{k}})^2 - \frac{\varrho_0\boldsymbol{k}^2}{m}(\delta S_{\boldsymbol{k}})^2\right]\,, \tag{41}$$

where V_Q denotes the quantisation volume. Note that, in contrast to Eq. (14), we did not neglect the quantum pressure term. In a straightforward manner, the phase and density fluctuations can be quantised via the introduction of creation ($\hat{a}^\dagger_{\boldsymbol{k}}$) and annihilation operators ($\hat{a}_{\boldsymbol{k}}$) which diagonalise the Hamiltonian. The quantised phase and density fluctuations read

$$\delta\hat{S}_{\boldsymbol{k}} = \sqrt{\frac{1}{V_Q\omega_{\boldsymbol{k}}}\left(g_{2\text{D}} + \frac{\boldsymbol{k}^2}{4m\varrho_0}\right)}\left(\hat{a}_{\boldsymbol{k}} + \hat{a}^\dagger_{\boldsymbol{k}}\right)\,, \tag{42}$$

$$\delta\hat{\varrho}_{\boldsymbol{k}} = i\sqrt{\frac{\omega_{\boldsymbol{k}}}{V_Q}\left(g_{2\text{D}} + \frac{\boldsymbol{k}^2}{4m\varrho_0}\right)^{-1}}\left(\hat{a}_{\boldsymbol{k}} - \hat{a}^\dagger_{\boldsymbol{k}}\right)\,, \tag{43}$$

with the well-known Bogoliubov dispersion relation

$$\omega_{\boldsymbol{k}}^2 = \mu\frac{\boldsymbol{k}^2}{m} + \frac{\boldsymbol{k}^4}{4m^2}\,, \tag{44}$$

where $\mu = g_{2\text{D}}\varrho_0^{2\text{D}}$ is the (initial) chemical potential.

For a perfectly scaling condensate, the two-point function in Eq. (39) yields the density-density correlations at all times, and in particular also the frozen correlations

(the density contrast) after horizon crossing. We can thus calculate the Fourier components

$$C_{2D}(\boldsymbol{\kappa}) = \int d^2\rho \frac{\langle \delta\hat{\varrho}(t,\mathbf{0})\delta\hat{\varrho}(t,\boldsymbol{\rho})\rangle}{\varrho_0^2(t)}\, e^{i\boldsymbol{\kappa}\cdot\boldsymbol{\rho}} = \frac{g_{2D}|\boldsymbol{\kappa}|}{\mu\sqrt{4m\mu+\boldsymbol{\kappa}^2}}\,. \tag{45}$$

in order to obtain the spectrum.

For large wavelengths, the spectrum is linear in the (co-moving) momentum $\boldsymbol{\kappa}$ in complete agreement to the curved space-time analogy. For large (co-moving) momenta $\boldsymbol{\kappa}$, where the dispersion relation (44) changes from linear to quadratic and the curved space-time analogy breaks down, the spectrum of the two-point correlation function approaches a constant value corresponding to a Dirac δ-contribution $C(\boldsymbol{\rho},\boldsymbol{\rho}') \propto \delta(\boldsymbol{\rho}-\boldsymbol{\rho}')+\ldots$. Interestingly, after subtracting this local term, we obtain a scale-invariant spectrum $C(\boldsymbol{\kappa}) \propto 1/\boldsymbol{\kappa}^2$ for large $\boldsymbol{\kappa}$ – in analogy to cosmic inflation, where the spectrum is scale-invariant, too (in 3+1 dimensions it is $1/k^3$). However, one should bear in mind that this scale-invariance occurs in a region where the curved space-time analogy breaks down.

For addressing the question of whether this effect (amplification of the initial quantum fluctuations of the phon modes) can be measured, we estimate the order of magnitude of the relative density-density correlations $C(\boldsymbol{\kappa})$ for $\boldsymbol{\rho} \neq \boldsymbol{\rho}'$. Since the two-dimensional coupling constant, g_{2D}, is proportional to the ratio of the scattering length, a_s, and the perpendicular extension of the condensate, a_z, we have typically

$$\frac{\langle \delta\hat{\varrho}(\boldsymbol{\rho})\delta\hat{\varrho}(\boldsymbol{\rho}')\rangle}{\varrho_0^2} = \mathcal{O}\left(\frac{a_s}{a_z}\right)\,. \tag{46}$$

Note, however, that this can only be seen as a rough estimate of the order of magnitude, since the two-point function is logarithmically divergent at small distances (where eventually the employed approximations break down). Remembering the conditions (8) in Section 2, we see that the relative density contrast must be significantly smaller than one – which was to be expected. However, the fluctuations could be on the percent-level (see Section 8) and thus might well be measurable.

7. Density-Density Correlations in 3D

For an expanding 3D condensate with quartic coupling the situation is more complicated, since it does not show the perfect scaling behaviour as in two dimensions. Therefore, it is not possible to infer the final (frozen) density (or phase) fluctuations from the initial state exactly as in the previous case. Fortunately, for long wavelengths, we may exploit the effective space-time analogy and derive the evolution using the tools known from cosmology. To this end, we transform onto co-moving coordinates and thus diagonalise the effective PGL-metric (19). Note that, in three (spatial) dimensions, the background density scales as $\rho(t) = \rho(0)/b^3(t)$. Assuming the background density to be smooth (Thomas-Fermi limit), the evolution equation for the scale factor – after switching off

the trap – reads, cf. [20]

$$\ddot{b}(t) = \frac{\omega_0^2}{b^4(t)} \,. \tag{47}$$

Similar to the previous case, the scale factor accelerates (the interaction energy is transformed into kinetic energy) after switching off the trap, but quickly assumes a linear behaviour, $b \propto t$ (for $t \gg 1/\omega_0$).

The following calculations are most conveniently performed using the laboratory time but co-moving spatial coordinates. One advantage of the analogy to curved space-times is the inherited freedom of choosing suitable coordinates, where the independence of the final result follows automatically from covariance. The equation of motion of the field modes $\phi_{\boldsymbol{\kappa}}$ with co-moving wavenumber $\boldsymbol{\kappa}$ assumes the form

$$\left(\frac{\partial^2}{\partial t^2} + 3\frac{\dot{b}}{b}\frac{\partial}{\partial t} + \frac{c_0^2\boldsymbol{\kappa}^2}{b^5}\right)\phi_{\boldsymbol{\kappa}} = 0 \,. \tag{48}$$

Initially, for the trapped condensate, the damping term $3\dot{\phi}_{\boldsymbol{\kappa}}\dot{b}/b$ vanishes and the modes oscillate freely. During the expansion, the third term decreases whilst the second one increases and eventually dominates the latter one – in complete analogy to an over-damped oscillator, the modes freeze in. Because most phonon modes have frequencies much larger than the trapping frequency (e.g., [23]), this horizon crossing and freezing process happens during the period of linear expansion $b(t) = \alpha t$ (with $\alpha \approx 0.82\omega_0$). In view of the adiabatic theorem, the initial vacuum state for these (initially) rapidly oscillating modes is preserved (adiabatic vacuum) during the initial non-linear expansion. Thus the equation of motion describing the freezing in of modes can be simplified considerably by inserting $b(t) = \alpha t$

$$\left(\frac{\partial^2}{\partial t^2} + 3\frac{1}{t}\frac{\partial}{\partial t} + \frac{c_0^2\boldsymbol{\kappa}^2}{\alpha^5 t^5}\right)\phi_{\boldsymbol{\kappa}} = 0 \tag{49}$$

and solved analytically in terms of Bessel functions [24]

$$\phi_{\boldsymbol{\kappa}} = C_{\boldsymbol{\kappa}}^{(1)}\frac{1}{t}H_{2/3}^{(1)}\left(\frac{2}{3}\frac{c_0\kappa}{\alpha^{5/2}}t^{-3/2}\right) + C_{\boldsymbol{\kappa}}^{(2)}\frac{1}{t}H_{2/3}^{(2)}\left(\frac{2}{3}\frac{c_0\kappa}{\alpha^{5/2}}t^{-3/2}\right) \,, \tag{50}$$

where $H_{2/3}^{(1,2)}$ denote the Hankel functions with the index $\nu = 2/3$ and $C_{\boldsymbol{\kappa}}^{(1,2)}$ are the corresponding integration constants. As one would expect, for early times, the modes $\phi_{\boldsymbol{\kappa}}$ oscillate $\dot{\phi}_{\boldsymbol{\kappa}}^{(1,2)} = \pm i\omega_{\rm ad}\phi_{\boldsymbol{\kappa}}^{(1,2)}$. It turns out that the Hankel functions $H_{2/3}^{(1,2)}$ have the proper asymptotic behaviour for early times such that the integration constants $C_{\boldsymbol{\kappa}}^{(1,2)}$ can be replaced by (time-independent) creation and annihilation operators (in analogy to the previous Section) associated to the initial adiabatic vacuum state for the quantised phonon modes $\hat{\phi}_{\boldsymbol{\kappa}}$ in the Heisenberg picture

$$C_{\boldsymbol{\kappa}}^{(1)} \to \sqrt{\frac{\pi}{6}\frac{g}{\alpha^3}\frac{2}{V_{\rm Q}}}\,\hat{a}_{\boldsymbol{\kappa}} \,, \tag{51}$$

where the pre-factor can be derived by imposing the canonical commutation relations for the conjugate variables ϕ and $\delta\varrho = \dot{\phi}/g$ obtained from the Lagrangian (18) for co-moving coordinates. Note that the argument of the Hankel functions coincides (up to a constant) with the proper time, $\tau \propto \frac{2}{3}\alpha^{-5/2}t^{-3/2}$, during the phase of linear expansion.

For late times $t \uparrow \infty$, the modes $\phi_{\boldsymbol{\kappa}}$ approach a constant value (due to horizon crossing and freezing) which can be derived inserting the asymptotic behaviour of the Hankel functions $H^{(1,2)}_{2/3}$ [24]. The frozen late-time $t \uparrow \infty$ expectation value for phase-phase correlations reads

$$\langle \hat{\phi}^2_{\boldsymbol{\kappa}} \rangle = \int d^3\rho \, \cos(\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) \langle \hat{\phi}(\boldsymbol{\rho}) \hat{\phi}(\boldsymbol{0}) \rangle = \frac{g}{6\pi} \left[\Gamma\left(2/3\right) \right]^2 \frac{\alpha^{1/3} 3^{4/3}}{c_0^{4/3}} \kappa^{-4/3} . \quad (52)$$

Employing Eq. (16), we can calculate the density fluctuations, $\delta\varrho = -\dot{\phi}/g$ (in co-moving coordinates, where $\boldsymbol{v}_0$ vanishes), and obtain the spectrum of the relative density-density correlation function at late times

$$C_{3\mathrm{D}}(\boldsymbol{\kappa}) = \int d^3\rho \cos(\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) \frac{\langle \delta\hat{\varrho}(\boldsymbol{\rho}) \delta\hat{\varrho}(\boldsymbol{0}) \rangle}{\varrho_0^2} = \frac{[\Gamma(1/3)]^2 3^{2/3}}{6\pi} \frac{\xi c_0^{1/3}}{\varrho_0 \alpha^{1/3}} \kappa^{4/3} , \quad (53)$$

where everything on the right-hand side (healing length etc.) refers to the initial state. Interestingly, one obtains the same spectra ($k^{-4/3}$ for the phase and $k^{+4/3}$ for the density fluctuations) for a condensate at rest after sweeping through the phase transition at $g = 0$ by means of a time-dependent coupling $g(t)$ [25].

In contrast to the quasi-2D case with perfect scaling, the above results rely on the curved space-time analogy and thus are only valid for wavelengths far above the healing length. Furthermore, in three dimensions, the healing length scales like $\xi(t) = \xi(0) b^{3/2}(t)$ and grows faster (again in contrast to the quasi-2D case) than the wavelengths $\lambda(t) = \lambda(0) b(t)$ in laboratory coordinates. In order to obtain a rough estimate of the maximum amplification effect, we insert the maximum co-moving wavenumber $\kappa = 1/\xi(\alpha/\omega_\xi)^{1/4}$ of the phonon modes whose wavelength exceeds the healing length during the relevant period of their evolutions (i.e., until horizon-crossing and freezing)

$$\begin{aligned} \kappa^3 C_{3\mathrm{D}}(\kappa) &= \frac{4\sqrt{\pi}[\Gamma(1/3)]^2 (\tilde{\alpha})^{3/4}}{3^{1/3}} \sqrt{a_s^3 \varrho_0} \left(\frac{\omega_\xi}{\omega_0} \right)^{-3/4} \\ &\approx 30.3 \sqrt{a_s^3 \varrho_0} \left(\frac{\omega_\xi}{\omega_0} \right)^{-3/4} . \end{aligned} \quad (54)$$

Here $\omega_\xi = c_0/\xi(0)$ denotes the (initial) frequency of the phonons with the (co-moving) wavenumber $1/\xi(0)$ and $\alpha = \tilde{\alpha}\omega_0$ where $\tilde{\alpha} \approx 0.82$ is a dimensionless constant. In contrast to the quasi-two-dimensional case, the size of the fluctuations now depends on the diluteness parameter $\sqrt{a_s^3 \varrho_0}$ (instead of the ratio a_s/a_z), which must be small for the employed approximations to apply $\sqrt{a_s^3 \varrho_0} \ll 1$. Furthermore, the ratio $\omega_\kappa/\omega_0 = c_0\kappa/\omega_0$ should be large $\omega_\kappa/\omega_0 \gg 1$ if we want a sufficiently long period of linear expansion $b(t) = \alpha t$. However, the smallness of these parameters can partly be compensated by the numerical pre-factor 30.3 such that the final effect can be on the percent-level (see the next Section).

The spectrum and size of the frozen-in correlations $C_{3\mathrm{D}}$ could be measured by obtaining a density map of a slice of the condensate (tomographic imaging). By making a projective (e.g., absorption) image instead, one would average over the fluctuations in the (spatial) direction of projection. While the spectrum remains unchanged, this

averaging yields an additional suppression by a factor of order ξ/l_z with l_z denoting the transversal (i.e., in direction of projection) extension of the condensate.

8. Summary

The objective was to investigate and to exploit the analogy between phonons in expanding Bose-Einstein condensates on the one hand and quantum fields in an expanding universe on the other hand. The advantages of this curved space-time analogy are two-fold: Firstly, it facilitates the application of all the concepts, tools, and effects known from general relativity (such as horizons) and thereby fosters a better understanding of condensed-matter phenomena in terms of universal geometrical concepts. Secondly, this analogy enables a theoretical and possibly experimental investigation of exotic quantum effects known from cosmology.

After a discussion of the effectively lower-dimensional behaviour of strongly constrained BECs in Section 2, the emergence of an metric governing the propagation of low-energy phonons is discussed in Section 3 for arbitrary power-law self-coupling in any dimension. In complete analogy to cosmology, the introduction of co-moving coordinates is advantageous for the description of many phenomena. It turns out that the full field operator (in a harmonic trapping potential) possesses perfectly scaling solutions exactly if the effective metric is flat in terms of the co-moving coordinates. In a freely expanding Bose-Einstein condensate, an effective sonic horizon is formed, i.e., two points at fixed (co-moving) spatial positions whose distance exceeds the horizon size cannot be connected anymore by (low-energy) phonons, cf. Section 5. The formation of this effective horizon implies the amplification of the quantum fluctuations (horizon-crossing and freezing) already known from cosmology, which has been calculated explicitely for quasi-2D and 3D condensates with quartic coupling in Sections 6 and 7.

In order to obtain an explicite estimate for the size of the derived effect, let us consider a condensate of 10^5 sodium atoms inside a highly anisotropic trap with the trapping frequencies $\omega_\perp/2\pi = 790\,\text{Hz}$ and $\omega_\parallel/2\pi = 10\,\text{Hz}$, cf. [26]. The thickness of the disk, $a_z = 0.746\,\mu\text{m}$, is smaller than the healing length, $\xi = 1.34\,\mu\text{m}$, but still much larger than the scattering length, $a_s = 2.8\,\text{nm}$, complying with the hierarchy of scales derived in Sec. 2. To estimate the order of magnitude of the density-density correlations, we consider a mode with (co-moving) wavenumber $\kappa = 2\pi/\xi$. For the relative density contrast inside a volume of size ξ^2 we obtain $C(\kappa)/\xi^2 = 1.79\,\%$. For the modes in the linear regime of the spectrum, horizon crossing and thus freezing in occurs shortly after the trap is switched off. Initially the horizon is at infinity, but very quickly settles at $r_{\text{horizon}}(t \uparrow \infty) = 3.28\,\mu\text{m}$. Hence already after one e-fold, when the radius of the atom cloud has increased by a factor $e \approx 2.7$, all modes with wavelength larger than ξ are frozen in and the fluctuations are transformed into a density contrast. With the correlation function depending on the ratio a_s/a_z, the effect can be enhanced by confining the condensate more tightly in the perpendicular direction. However, if a_s/a_z is not small, the condensate is no longer described by the usual Gross-

Pitaevskiĭ Lagrangian and our analysis does not apply anymore in this form (e.g., the perfect scaling breaks down). For such a case, the (initial) correlation spectrum (with appropriate non-quartic coupling, cf. [12, 27]) was calculated in Ref. [27] and differs from our result in Eq. (45).

As an example for the 3D case, let us consider ^{87}Rb atoms inside a spherically symmetric trap. For a condensate consisting of 10^7 atoms and a trapping frequency $\omega_0/2\pi = 200\,\mathrm{Hz}$, which are potentially realisable parameters, cf. [28], the Thomas-Fermi radius is $R = 12.2\,\mu\mathrm{m}$. The minimal frequency of phonons $\omega_{\min} = 2\pi c_s/R = 5582\,\mathrm{Hz}$ is well above the trapping frequency ω_0. Hence the freezing of modes takes place during the period of linear expansion and the approximation $b(t) = \alpha t$ in the evolution equation of the phonon modes is justified. A rough estimate of the maximum effect according to Eq. (54) yields

$$\frac{\langle \delta\hat{\varrho}(\boldsymbol{\rho})\delta\hat{\varrho}(\boldsymbol{\rho'})\rangle}{\varrho_0^2} = \mathcal{O}\,(2\,\%) \tag{55}$$

i.e., a potentially measurable effect

So far, we have assumed zero temperature. Of course, initial thermal fluctuations would – provided that there is no thermalisation during the expansion of the condensate – also be amplified by basically the same mechanism as the quantum fluctuations. In order to ensure that the quantum fluctuations are larger than the (initial) thermal fluctuations, the temperature must be small enough such that the thermal occupation of the phonon modes under consideration is negligible. For the parameters used above, the relevant temperature scales are $T(1/\xi) = 0.1\,\mathrm{nK}$ for the Q2D and $T(\kappa) = 3.9\,\mathrm{nK}$ for the 3D condensate, respectively. By increasing the particle number and/or the trapping frequency, the requirements regarding the experimental temperature can be relieved. (Another way to discriminate between thermal and quantum fluctuations is the spectrum.)

In summary, expanding Bose-Einstein condensates facilitate the experimental simulation of exotic effects of quantum fields in curved space-times. On the other hand, the amplification mechanism under consideration sets the ultimate quantum limit of accuracy for time-of-flight experiments: No matter how smooth the initial cloud can be prepared, the frozen and amplified quantum ground-state fluctuations generate noticeable density perturbations.

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